Mean Values of $\zeta'/\zeta(s)$, Correlations of Zeros, and the Distribution of Almost Primes

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The Riemann zeta function $\zeta(s)$

1. On $\mathrm{Re}\, s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}.$$

2. Analytic continuation to $\mathbb{C} \setminus \{1\}$.
3. Functional equation $\xi(s) := \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \xi(1-s)$.
4. No zeros outside of the critical strip $0 < \mathrm{Re}\, s < 1$ except trivial zeros $-2, -4, -6, \ldots$.

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Riemann Hypothesis

All the nontrivial zeros of $\zeta(s)$ are on the critical line $\mathrm{Re}\, s = 1/2$.

In this talk, we assume RH!

Pair Correlation of zeros of $\zeta(s)$

Assume RH. Define

$$F(\alpha, T) = (\frac{T}{2\pi}\log T)^{-1} \sum_{0<\gamma,\gamma'<T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'),$$

where $1/2 + i\gamma$ and $1/2 + i\gamma'$ are zeros of $\zeta(s)$
and $w(u) = 4/(4+u^2)$ is a weight function.

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1. $F(\alpha, T)$ is even.
2. $F(\alpha, T)$ is nonnegative.
   Since $w(\gamma-\gamma') = 2/\pi \int_{-\infty}^{\infty} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)}$, we see that

$$F(\alpha, T) = \frac{4}{T\log T}\int_{-\infty}^{\infty} \left| \sum_{0<\gamma<T} \frac{T^{i\alpha\gamma}}{1+(t-\gamma)^2} \right|^2 dt \geq 0.$$

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Theorem[Montgomery]

Assume RH. For $|\alpha| \leq 1$, we have
$F(\alpha, T) = |\alpha| + o(1) + T^{-2|\alpha|}\log T(1+o(1))$.

Conjecture 1

$F(\alpha, T) = 1 + o(1)$ for $\alpha > 1$.

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- Difficulty of Conjecture 1

$F(\alpha, T)$

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$$G(\alpha, T) = (\frac{T}{2\pi}\log T)^{-1} \sum_{0<\gamma,\gamma'<T} \left(\frac{\sin \frac{\alpha}{2}(\gamma-\gamma')\log T}{\frac{\alpha}{2}(\gamma-\gamma')\log T} \right)^2 w(\gamma-\gamma').$$

Theorem[Montgomery]

For $0 < \alpha \leq 1$, we have $G(\alpha, T) \sim \frac{1}{\alpha} + \frac{\alpha}{3}$.

Theorem[Goldston, Gonek] 1990

For $a > 0$, β real, and $T \geq 2$,
$a\left(a - \frac{1}{2}G\left(\frac{a}{2}, T\right)\right) \leq \int_\beta^{\beta+a} F(\alpha, T)d\alpha \leq a\left(G(a, T) + \frac{1}{2}G\left(\frac{a}{2}, T\right)\right).$

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As a consequence, $\int_{\beta}^{\beta+1} F(\alpha, T)d\alpha$ is bounded.

Sketched proof of Montgomery's Theorem

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Thus, we have

$$\sum_{0<\gamma,\gamma'<T} r((\gamma-\gamma')(2\pi)^{-1}\log T) w(\gamma-\gamma')$$
$$= \sum_{0<\gamma,\gamma'<T} \left(\int_{-\infty}^{\infty} \hat{r}(v) T^{i(\gamma-\gamma')v} dv\right) w(\gamma-\gamma')$$
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$= (2\pi)^{-1}T\log T\int_{-\infty}^{\infty}F(v,T)\hat{r}(v)dv$.
Choose $r(u) = ((\sin\pi\alpha u)/\pi\alpha u)^2$, then $(LHS) = (\frac{T}{2\pi}\log T)G(\alpha, T)$ and

$$\int_{-\infty}^{\infty}F(v,T)\hat{r}(v)dv = \frac{1}{v^2}\int_{-\alpha}^{\alpha}(\alpha-|v|)F(v,T)dv.$$

Goldston, Gonek and Montgomery's work

Let $\psi(x) = \sum_{n \le x} \Lambda(n)$, where $\Lambda(n) = \log p$ if n is a prime power p^k and $\Lambda(n) = 0$ otherwise. Define

$$I(\sigma, T) = \int_1^T \left|\zeta'/\zeta(\sigma + it)\right|^2 dt$$

$$P(\beta, T) = \int_1^\infty (\psi(x + x/T) - \psi(x) - x/T)^2 x^{-2-2\beta} dx$$

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Note that PNT says $\psi(x) \sim x$. Assume RH and suppose $A > 0$ is fixed. If there exists a number $f(A)$ such that one of the following asymptotic formulas is true as $T \to \infty$, then all of them are true:

$$I\left(\frac{1}{2} + \frac{A}{\log T}; \ T\right) \sim f(A)\, T \log^2 T,$$

$$\int_{0^+}^\infty F(\alpha; \ T) e^{-2A\alpha} d\alpha \sim f(A),$$

$$P\left(\frac{A}{\log T}; \ T\right) \sim f(A) \frac{\log^2 T}{T}.$$

Higher Analogue of I

$N = J + K \geq 2$, $J \geq 0, K \geq 1$.
$\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$, $\varepsilon_j = 1$ for $j \leq J$, $\varepsilon_j = -1$ for $J < j \leq J + K = N$.
$\mathbf{a} = (a_1, a_2, \ldots, a_N)$ with $a_n > 0$ and $a_n \approx 1/\log T$ for $1 \leq n \leq N$.
Here $a_n \approx 1/\log T$ means there exist constants $0 < A_n \leq A_n^{'}$ such that $A_n/\log T \leq |a_n| \leq A_n^{'}/\log T$.
Our generalization of the mean value $I(\sigma; T)$ is

$$I(\sigma, \mathbf{a}, \varepsilon; T) = \int_0^T \prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + i\varepsilon_n t)\, dt.$$

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When $N = 2$, $\boldsymbol{\varepsilon} = (1, -1)$ and $\mathbf{a} = (a, a)$, we have

$$I(\sigma, \mathbf{a}, \boldsymbol{\varepsilon}; T) = \int_0^T \left| \frac{\zeta'}{\zeta}(\sigma + a + it) \right|^2 dt.$$

Higher Analogue of F

Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_{N-1})$ with $\alpha_n \in \mathbb{R}$.
Our generalization of $F(\alpha; T)$ is

$$F(\boldsymbol{\alpha}; T) = N(T)^{-1} \sum_{0<\gamma_1,...,\gamma_N<T} T^{\,i \sum_{n<N} \alpha_n(\gamma_n - \gamma_N)} w(\gamma_1 - \gamma_N, ..., \gamma_{N-1} - \gamma_N)$$

where $N(T)$ is the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma < T$ and $w(x_1, \ldots, x_{N-1}) = \prod_{n=1}^{N-1} \frac{4}{4+x_n^2}$ is a weight function. Note that $N(T) \sim \frac{T}{2\pi} \log T$.

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Let $\mathbf{e}_n = (0, \ldots, 1, \ldots, 0)$ for $1 \leq n < N$ and $\mathbf{e}_N = (-1, \ldots, -1)$. Then $\boldsymbol{\alpha} \cdot \mathbf{e}_n = \alpha_n,\ 1 \leq n < N$ and $\boldsymbol{\alpha} \cdot \mathbf{e}_N = -\alpha_1 - \cdots - \alpha_{N-1}$ and

$$F(\boldsymbol{\alpha}; T) = N(T)^{-1} \sum_{0<\gamma_1,...,\gamma_N<T} T^{\,i\sum_{n=1}^{N} (\boldsymbol{\alpha}\cdot\mathbf{e}_n)\gamma_n} w(\gamma_1 - \gamma_N, ..., \gamma_{N-1} - \gamma_N).$$

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Note that $\boldsymbol{\alpha}\cdot\mathbf{e}_n = \alpha_n,\ 1 \le n < N$ and $\boldsymbol{\alpha}\cdot\mathbf{e}_N = -\alpha_1 - \cdots - \alpha_{N-1}$.
When $\boldsymbol{\alpha}\cdot\mathbf{e}_n = 0$ for some $1 \le n \le N$, there is no cancelation on the sum over γ_n.
Thus, we expect that $F(\boldsymbol{\alpha}; T)$ has *Spike* along the hyperplanes $\boldsymbol{\alpha}\cdot\mathbf{e}_n = 0,\ 1 \le n \le N$.
We write $F^*(\boldsymbol{\alpha}; T)$ for the part of $F(\boldsymbol{\alpha}; T)$ that is supported outside the spikes from the lower correlation terms.

Hypothesis AC on $F(\alpha; T)$

$F^*(\alpha; T)$: the part of $F(\alpha; T)$ supported outside the spikes from the lower correlation terms.

Hypothesis AC

We have

$$\int_{x_1}^{x_1+1} \cdots \int_{x_{N-1}}^{x_{N-1}+1} |F^*(\alpha;\ T)|\, d\alpha \ll 1$$

uniformly for $(x_1, x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$.

That is, averages of F^* is bounded. When $N = 2$, Hypothesis AC is known.

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Let $F_*(\alpha; T) = F(\alpha; T) - F^*(\alpha; T)$. How small F_* is?

Inside the spikes ($N = 3$)

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. Suppose $\alpha_2 = 0$. Then

$$F(\alpha_1, 0; T) = N(T)^{-1} \sum_{0<\gamma_1,\gamma_2,\gamma_3<T} T^{i\alpha_1(\gamma_1-\gamma_3)} w(\gamma_1 - \gamma_3, \gamma_2 - \gamma_3).$$

Summing over γ_2, we expect that

$$F(\alpha_1, 0; T) \sim \frac{\log T}{N(T)} \sum_{0<\gamma_1,\gamma_3<T} T^{i\alpha_1(\gamma_1-\gamma_3)} w(\gamma_1 - \gamma_3) = (\log T) F(\alpha_1; T).$$

Since the "spike" term in $F(\alpha_2; T)$ is $(1 + o(1)) T^{-2|\alpha_2|} \log T$,
we expect that $F(\alpha_1, \alpha_2; T)$ is approximately $T^{-2|\alpha_2|} \log T \; F(\alpha_1; T)$
when $|\alpha_2| \leq \log\log T / (2 \log T)$.
The same argument applies when α_1 or $\alpha_1 + \alpha_2$ is near 0.

Hypothesis LC on $F(\alpha; T)$

More generally, $F(\alpha; T)$ degenerates into a lower level sum on the set $S = \bigcup_{n=1}^{N} S_n$, where $S_n = \left\{\alpha \in \mathbb{R}^{N-1} \mid \alpha \cdot \mathbf{e}_n = 0\right\}$ for $1 \le n \le N$. Define $\eta_n = \{\mathbf{t} \in \mathbb{R}^{N-1} \mid |\mathbf{t} - \mathbf{y}| < \log\log T/(2\log T)$ for some $\mathbf{y} \in S_n\}$ and $\eta = \bigcup_{n=1}^{N} \eta_n$. Then

Hypothesis LC

$F(\alpha; T) = F_*(\alpha; T) + F^*(\alpha; T)$

1. $F_*(\alpha; T)$ is supported on η and $F_*(\alpha; T) \ll |F(\widetilde{\alpha}_n; T)| T^{-2|\alpha_n|} \log T$ if $\alpha \in \eta_n$ for some $1 \le n \le N$.
2. For any fixed $K > 0$, $F^*(\alpha; T)$ is bounded on the $(N-1)$-dimensional cube $[-K, K]^{N-1}$, as $T \to \infty$.

$\widetilde{\alpha}_n$ is obtained from α by deleting α_n for $n < N$. If $n = N$, delete any one of $\alpha_1, \ldots, \alpha_{N-1}$.

Higher Analogue of P

To define our analogue of $P(\beta; T)$ let $\mathbf{b} = (b_1, b_2, \dots, b_L)$ with $b_l > 0$ for $1 \leq l \leq L$. We define $\Lambda_{\mathbf{b}}(n)$ by

$$\prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s + b_l) = (-1)^L \sum_n \frac{\Lambda_{\mathbf{b}}(n)}{n^s},$$

where $\sigma > 1$. Then

$$\Lambda_{\mathbf{b}}(n) = \sum_{p_1^{\nu_1} p_2^{\nu_2} \cdots p_L^{\nu_L} = n} \frac{\log p_1 \cdots \log p_L}{p_1^{b_1 \nu_1} p_2^{b_2 \nu_2} \cdots p_L^{b_L \nu_L}}$$

Thus $\Lambda_{\mathbf{b}}(n)$ is supported on those positive integers n that are representable as a product of L, not necessarily distinct, prime powers.

Higher Analogue of P

We define $R_{\mathbf{b}}(x)$ to be the sum of the residues of

$$\prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s+b_l)\ \frac{x^s}{s}$$

at the points $s = 1 - b_l$. Next we set

$$\Psi_{\mathbf{b}}(x) = (-1)^L \sideset{}{'}\sum_{n \le x} \Lambda_{\mathbf{b}}(n),$$

where the prime on the sum indicates that the term $\Lambda_{\mathbf{b}}(x)$ is counted with weight $1/2$. We also write

$$\Delta_{\mathbf{b}}(x) = \Psi_{\mathbf{b}}(x) - R_{\mathbf{b}}(x).$$

Thus, $\Delta_{\mathbf{b}}$ measures the difference between $\Psi_{\mathbf{b}}(x)$ and its expected value.

Higher Analogue of P

Now let $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ with $a_n > 0$ and $a_n \approx 1/\log T$ as before. Also let $\beta > 0$ and $1 \le J < N$. Writing $\mathbf{a}_J = (a_1, a_2, \ldots, a_J)$ and $\mathbf{a}'_J = (a_{J+1}, a_{J+2}, \ldots, a_N)$, we set

$$P(\beta, \mathbf{a}, J;\ T) =$$

$$= \int_1^\infty \left(\Delta_{\mathbf{a}_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}_J}(x)\right)\left(\Delta_{\mathbf{a}'_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}'_J}(x)\right)\frac{dx}{x^{2+2\beta}}\ .$$

This is our analogue of $P(\beta; T)$.

Equivalence between I and F

Theorem 1

Assume RH, Hypothesis AC, and Hypothesis LC. Let $\mathbf{a} = (a_1, \ldots, a_N)$, where the $a_n \approx 1/\log T$ and are positive, and let $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$ consist of $J \geq 0$ ones followed by $K \geq 1$ negative ones. Then

$$I\Big(\frac{1}{2}, \mathbf{a}, \boldsymbol{\varepsilon}; T\Big) = T \log^N T \int\limits_{U_{N,\boldsymbol{\varepsilon}}} F^*(\boldsymbol{\alpha}; T)\, T^{-\sum_{n\leq N} a_n \varepsilon_n \alpha_n}\, d\boldsymbol{\alpha} + o(T \log^N T),$$

where $U_{N,\boldsymbol{\varepsilon}} = \{(\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{R}^{N-1} | \epsilon_1 \alpha_1 > 0, \ldots, \epsilon_N \alpha_N > 0\}$ and $\alpha_N = -\sum_{n<N} \alpha_n$.

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When $N = 2$, $\varepsilon = (1, -1)$, $\mathbf{a} = (A/\log T, A/\log T)$ and $\alpha_2 = -\alpha_1$, we have $U_{2,\varepsilon} = \{\alpha_1 \in \mathbb{R} | \alpha_1 > 0\}$, $\sum_{n\leq 2} a_n \epsilon_n \alpha_n = 2A\alpha_1/\log T$ and

$$I(1/2, \mathbf{a}, \varepsilon; T) \sim T(\log T)^2 \int_0^\infty F^*(\alpha_1; T) e^{-2A\alpha_1} d\alpha_1.$$

Equivalence between I and F

Corollary 1

With the same hypotheses as in Theorem 1, we have

$$I\left(\frac{1}{2}, \mathbf{a}, \varepsilon; T\right) \ll T \log^N T.$$

Equivalence between I and P

Theorem 2

Assume RH and let $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ with $a_n = A_n/\log T$ and $A_n > 0$ for $1 \leq n \leq N$. Also let $1 \leq J < N$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$, where $\varepsilon_1, \ldots, \varepsilon_J$ are all one, and $\varepsilon_{J+1}, \ldots, \varepsilon_N$ are all negative one. Then for $1/2 \leq \sigma \leq 9/10$ we have

$$\int_{-\infty}^{\infty} \left(\prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + i\varepsilon_n t) \right) \left(\frac{\sin t/2T}{t} \right)^2 dt$$
$$= \frac{\pi}{2} P\Big(\sigma - \frac{1}{2}, \mathbf{a}, J;\ T\Big) + O\left(\frac{\log^{2N+1} T}{T^2} \right).$$

The constant implied by the O-term depends on $A_1, \ldots, A_N$ but not on σ, J, or T.

Equivalence between I and P

Theorem 3

Assume RH, Hypothesis AC, and Hypothesis LC. Suppose that C is fixed and positive, and that $\mathbf{a} = (a_1, \ldots, a_N)$ with $a_n = A_n/\log T$ and each A_n fixed and positive. Define

$$I_{\pm}(\sigma, \mathbf{a}, \varepsilon; T) = \int_{-T}^{T} \prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + i\varepsilon_n t)\, dt.$$

If there exists a number $f(C, \mathbf{A}, J)$ such that one of the following asymptotic formulas holds, then the other also holds:

$$I_{\pm}\left(\frac{1}{2} + \frac{C}{\log T}, \frac{\mathbf{A}}{\log T}, \varepsilon;\ T\right) \sim f(C, \mathbf{A}, J)\, T \log^N T,$$

$$P\left(\frac{C}{\log T}, \frac{\mathbf{A}}{\log T}, J;\ T\right) \sim f(C, \mathbf{A}, J)\, \frac{\log^N T}{2T}.$$

Recall Theorem 1 ($I \leftrightarrow F$)

Theorem 1

Assume RH, Hypothesis AC, and Hypothesis LC. Let $\mathbf{a} = (a_1, \ldots, a_N)$, where the $a_n \approx 1/\log T$ and are positive, and let $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$ consist of $J \geq 0$ ones followed by $K \geq 1$ negative ones. Then

$$I\Big(\frac{1}{2}, \mathbf{a}, \varepsilon; T\Big) = T \log^N T \int\limits_{U_{N,\varepsilon}} F^*(\boldsymbol{\alpha}; T)\, T^{-\sum_{n \leq N} a_n \varepsilon_n \alpha_n}\, d\boldsymbol{\alpha} + o(T \log^N T),$$

where $U_{N,\varepsilon} = \{(\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{R}^{N-1} | \epsilon_1 \alpha_1 > 0, \ldots, \epsilon_N \alpha_N > 0\}$ and $\alpha_N = -\sum_{n<N} \alpha_n$.

Proof of Theorem 1

Assume RH.

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-\delta n} + \sum_{\rho} \delta^{s-\rho}\Gamma(\rho - s) + O(\delta^{\sigma-1/4} \log t)$$

uniformly for $e^{-\sqrt{t}} \le \delta \le 1$ and $\frac{1}{2} \le \sigma \le \frac{9}{8}$.

Lemma 1

Assume RH. Let $X = (\log T)^{4/3}$, $a \approx 1/\log T$ with $a > 0$, and $\varepsilon = \pm 1$. Then for $|t| < T$ we have

$$\frac{\zeta'}{\zeta}(\tfrac{1}{2} + a + i\varepsilon t) = -\sum_{\gamma} R(-a + i\varepsilon(\gamma - t)) + O(X^{1/2}),$$

where $R(z) = X^z \Gamma(z)$.

Proof of Theorem 1

Recall the definition of I :

$$I(\sigma, \mathbf{a}, \varepsilon; T) = \int_0^T \prod_{n=1}^N \frac{\zeta'}{\zeta}(\sigma + a_n + i\varepsilon_n t)\, dt.$$

By Lemma 1, we have

$$I\Big(\frac{1}{2}, \mathbf{a}, \varepsilon; T\Big) = (-1)^N M(\mathbf{a}, \varepsilon; T) + O(T(\log T)^{N-1/3}),$$

where

$$M(\mathbf{a}, \varepsilon;\ T) \ = \ \int_0^T \prod_{n=1}^N \Big(\sum_{\gamma_n} R(-a_n + i\varepsilon_n(\gamma_n - t)) \Big)\, dt.$$

Truncate the sums of γ_n's, and extend the integral from $-\infty$ to ∞. Then

$$M(\mathbf{a}, \varepsilon; T) = \sum_{0<\gamma_1,\dots,\gamma_N<T} \int_{-\infty}^{\infty} \prod_{n=1}^N R(-a_n + i\varepsilon_n(\gamma_n - t))\, dt + O((\log T)^B)$$

Proof of Theorem 1

Change the variable $t \to t + \gamma_N$, then

$$M(\mathbf{a}, \varepsilon; T) = \sum_{0<\gamma_1,\dots,\gamma_N<T} \int_{-\infty}^{\infty} \prod_{n=1}^{N} R(-a_n + i\varepsilon_n(\gamma_n - \gamma_N - t))\, dt$$
$$+ O((\log T)^B)$$
$$= \sum_{0<\gamma_1,\dots,\gamma_N<T} \mathcal{R}\big(\widetilde{\gamma}_1 - \widetilde{\gamma}_N, \dots, \widetilde{\gamma}_{N-1} - \widetilde{\gamma}_N\big) + O((\log T)^B),$$

where $L = (1/2\pi)\log T$, $\widetilde{\gamma}_j = \gamma_j L$ and

$$\mathcal{R}(\mathbf{u}) = \int_{-\infty}^{\infty} \prod_{n=1}^{N} R(-a_n + i\varepsilon_n(u_n/L - t))\, dt$$

for $\mathbf{u} = (u_1, \dots, u_{N-1})$ and $u_N = 0$.

Proof of Theorem 1

We define $r(\mathbf{u}) = \mathcal{R}(\mathbf{u})w(\mathbf{u}/L)^{-1}$ and $w(\mathbf{x}) = \prod_{n=1}^{N-1} \frac{4}{4+x_n^2}$. Then

$$M(\mathbf{a}, \varepsilon; T) \sim \sum_{0<\gamma_1,\ldots,\gamma_N<T} r(\widetilde{\gamma}_1 - \widetilde{\gamma}_N, \ldots, \widetilde{\gamma}_{N-1} - \widetilde{\gamma}_N) w(\gamma_1 - \gamma_N, \ldots, \gamma_{N-1} - \gamma_N).$$

Since

$$\begin{aligned} r(\widetilde{\gamma}_1 - \widetilde{\gamma}_N, \ldots, \widetilde{\gamma}_{N-1} - \widetilde{\gamma}_N) &= \int_{\mathbb{R}^{N-1}} \widehat{r}(\alpha) e^{2\pi i \sum_{n<N} \alpha_n (\widetilde{\gamma}_n - \widetilde{\gamma}_N)} d\alpha \\ &= \int_{\mathbb{R}^{N-1}} \widehat{r}(\alpha) T^{i \sum_{n<N} \alpha_n (\gamma_n - \gamma_N)} d\alpha, \end{aligned}$$

we have

$$M(\mathbf{a}, \varepsilon; T) \sim N(T) \int_{\mathbb{R}^{N-1}} F(\alpha; T)\, \widehat{r}(\alpha)\, d\alpha.$$

Proof of Theorem 1

$$I\Big(\frac{1}{2}, \mathbf{a}, \varepsilon;\ T\Big) = (-1)^N N(T) \int\limits_{\mathbb{R}^{N-1}} F(\alpha; T)\, \widehat{r}(\alpha)\, d\alpha + O(T\, L^{N-1/3})$$

Our next task is to find a useful expression for $\widehat{r}(\alpha)$.

Proof of Theorem 1

$$I\left(\frac{1}{2}, \mathbf{a}, \varepsilon;\ T\right) = (-1)^N N(T) \int\limits_{\mathbb{R}^{N-1}} F(\boldsymbol{\alpha}; T)\, \widehat{r}(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} + O(T\, L^{N-1/3})$$

Our next task is to find a useful expression for $\widehat{r}(\boldsymbol{\alpha})$.

$$\begin{aligned}
\widehat{r}(\boldsymbol{\alpha}) &= \int\limits_{\mathbb{R}^{N-1}} r(\mathbf{u}) e^{-2\pi i \boldsymbol{\alpha}\cdot\mathbf{u}}\, d\mathbf{u} \\
&= \int\limits_{\mathbb{R}^{N-1}} \mathcal{R}(\mathbf{u}) \prod_{n=1}^{N-1} \left(1 + \frac{u_n^2}{4L^2}\right) e^{-2\pi i \boldsymbol{\alpha}\cdot\mathbf{u}}\, d\mathbf{u} \\
&= \int\limits_{\mathbb{R}^{N-1}} \prod_{n=1}^{N-1} \left(1 - \frac{1}{16\pi^2 L^2} \frac{\partial^2}{\partial \alpha_n^2}\right) \mathcal{R}(\mathbf{u}) e^{-2\pi i \boldsymbol{\alpha}\cdot\mathbf{u}}\, d\mathbf{u} \\
&= \prod_{n=1}^{N-1} \left(1 - \frac{1}{16\pi^2 L^2} \frac{\partial^2}{\partial \alpha_n^2}\right) \widehat{\mathcal{R}}(\boldsymbol{\alpha}).
\end{aligned}$$

Proof of Theorem 1

$$\widehat{\mathcal{R}}(\boldsymbol{\alpha}) = \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N} R(-a_n + i\varepsilon_n (u_n/L - t))\, e^{-2\pi i \alpha_n u_n} \right) dt\, d\mathbf{u}$$

$$= \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\infty} R(-a_N - i\varepsilon_N t) \left(\prod_{n=1}^{N-1} R(-a_n + i\varepsilon_n u_n/L)\, e^{-2\pi i \alpha_n (u_n + Lt)} \right) dt\, d\mathbf{u}$$

by the substitutions $u_n \to u_n + tL$ for $n < N$. Since $R(z) = X^z \Gamma(z)$, we can apply Lemma 2 to above equation.

Lemma 2

Let $0 < a < 1$, $A \in \mathbb{R}$, and $\varepsilon = \pm 1$. Then

$$\int_{-\infty}^{\infty} e^{iA\xi}\, \Gamma(-a + i\varepsilon\xi)\, d\xi = 2\pi\, e^{\varepsilon a A} \left(e^{-e^{-\varepsilon A}} - 1 \right).$$

Recall Theorem 2 ($I \leftrightarrow P$)

Theorem 2

Assume RH and let $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ with $a_n = A_n/\log T$ and $A_n > 0$ for $1 \le n \le N$. Also let $1 \le J < N$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$, where $\varepsilon_1, \ldots, \varepsilon_J$ are all one, and $\varepsilon_{J+1}, \ldots, \varepsilon_N$ are all negative one. Then for $1/2 \le \sigma \le 9/10$ we have

$$\int_{-\infty}^{\infty} \left(\prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + i\varepsilon_n t) \right) \left(\frac{\sin t/2T}{t} \right)^2 dt$$

$$= \frac{\pi}{2} P\Big(\sigma - \frac{1}{2}, \mathbf{a}, J;\ T\Big) + O\Big(\frac{\log^{2N+1} T}{T^2} \Big).$$

The constant implied by the O-term depends on $A_1, \ldots, A_N$ but not on σ, J, or T.

Proof of Theorem 2

Lemma 3

Assume RH. Suppose that $|b_l| < \frac{1}{10}$ with $\mathrm{Re}\, b_l > 0$. Then for $\frac{1}{2} \leq \sigma_0 \leq \frac{9}{10}$,

$$\Psi_{\mathbf{b}}(x) = (-1)^N {\sum_{n \leq x}}' \Lambda_{\mathbf{b}}(n) = R_{\mathbf{b}}(x) + \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s + b_l) \frac{x^s}{s} ds,$$

where $R_{\mathbf{b}}(x)$ is the sum of the residues of

$$\prod_{l=1}^{L} \frac{\zeta'}{\zeta}(s + b_l)\, \frac{x^s}{s}$$

at the points $s = 1 - b_l$.

Lemma 3 holds by Perron's formula.

Proof of Theorem 2

Recalling that $\Delta_{\mathbf{a}_J}(x) = (-1)^N \sum'_{n \le x} \Lambda_{\mathbf{a}_J}(n) - R_{\mathbf{a}_J}(x)$, we see from Lemma 3 that

$$\frac{\Delta_{\mathbf{a}_J}(e^{\tau+\delta}) - \Delta_{\mathbf{a}_J}(e^{\tau})}{e^{\sigma\tau}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{j=1}^{J} \frac{\zeta'}{\zeta}(\sigma + a_j + it) \left(\frac{e^{\delta(\sigma+it)} - 1}{\sigma + it} \right) e^{-2\pi it(-\tau/2\pi)} dt$$

for $\frac{1}{2} \le \sigma \le \frac{9}{10}$. This expresses the left-hand side as a Fourier transform.

Proof of Theorem 2

We use Plancherel's formula in the form

$$\int_{-\infty}^{\infty} \widehat{f}(\tau)\widehat{g}(\tau)d\tau = \int_{-\infty}^{\infty} f(t)g(-t)dt,$$

where

$$\widehat{f}(\tau) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\tau}dt$$

and similarly for $\widehat{g}$. Then we obtain

$$\int_{-\infty}^{\infty} \left(\Delta_{\mathbf{a}_J}(e^{\tau+\delta}) - \Delta_{\mathbf{a}_J}(e^{\tau})\right)\left(\Delta_{\mathbf{a}'_J}(e^{\tau+\delta}) - \Delta_{\mathbf{a}'_J}(e^{\tau})\right)e^{-2\sigma\tau}d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + i\varepsilon_n t) \left|\frac{e^{\delta(\sigma+it)} - 1}{\sigma + it}\right|^2 dt.$$

Compare it with the definition $P(\beta, \mathbf{a}, J;\ T) =$
$\int_1^{\infty} \left(\Delta_{\mathbf{a}_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}_J}(x)\right)\left(\Delta_{\mathbf{a}'_J}\left(x + \frac{x}{T}\right) - \Delta_{\mathbf{a}'_J}(x)\right)\frac{dx}{x^{2+2\beta}}$.

Recall Theorem 3 ($I \leftrightarrow P$)

Theorem 3

Assume RH, Hypothesis AC, and Hypothesis LC. Suppose that C is fixed and positive, and that $\mathbf{a} = (a_1, \ldots, a_N)$ with $a_n = A_n / \log T$ and each A_n fixed and positive. Define

$$I_{\pm}(\sigma, \mathbf{a}, \varepsilon; T) = \int_{-T}^{T} \prod_{n=1}^{N} \frac{\zeta'}{\zeta}(\sigma + a_n + i\varepsilon_n t)\, dt.$$

If there exists a number $f(C, \mathbf{A}, J)$ such that one of the following asymptotic formulas holds, then the other also holds:

$$I_{\pm}\left(\frac{1}{2} + \frac{C}{\log T}, \frac{\mathbf{A}}{\log T}, \varepsilon;\ T\right) \sim f(C, \mathbf{A}, J)\, T \log^N T,$$

$$P\left(\frac{C}{\log T}, \frac{\mathbf{A}}{\log T}, J;\ T\right) \sim f(C, \mathbf{A}, J)\, \frac{\log^N T}{2T}.$$

Proof of Theorem 3

The first asymptotic formula of Theorem 3 is

$$I_{\pm}\left(\frac{1}{2}+\frac{C}{\log T},\frac{\mathbf{A}}{\log T},\varepsilon;\ T\right)=\int_{-T}^{T}\prod_{n=1}^{N}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{C+A_n}{\log T}+i\varepsilon_n t\right)\,dt$$

$$=2\int_{0}^{T}\mathrm{Re}\prod_{n=1}^{N}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{C+A_n}{\log T}+i\varepsilon_n t\right)\,dt$$

$$\sim f(C,\mathbf{A},J)\,T\log^N T.$$

Define

$$g(t,\eta)=2\mathrm{Re}\prod_{n=1}^{N}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{C+A_n}{\log\eta}+i\varepsilon_n t\right)\Big/\left(f(C,\mathbf{A},J)\log^N\eta\right).$$

Then it is

$$\int_0^T g(t,T)dt\sim T.$$

Proof of Theorem 3

The second asymptotic formula of Theorem 3 is, by Theorem 2,

$$P\left(\frac{C}{\log T}, \frac{\mathbf{A}}{\log T}, J;\ T\right)$$

$$\sim \int_{-\infty}^{\infty} \left(\prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{C + A_n}{\log T} + i\varepsilon_n t\right)\right)\left(\frac{\sin t/2T}{t}\right)^2 dt$$

$$= 2\int_{0}^{\infty} \left(\mathrm{Re} \prod_{n=1}^{N} \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{C + A_n}{\log T} + i\varepsilon_n t\right)\right)\left(\frac{\sin t/2T}{t}\right)^2 dt$$

$$\sim f(C, \mathbf{A}, J)\,\frac{\log^N T}{2T}$$

and it is equivalent to

$$\int_0^{\infty} g(t, T)\left(\frac{\sin t/2T}{t}\right)^2 dt \sim \frac{\pi}{2}\frac{1}{2T}.$$

Proof of Theorem 3

Therefore, Theorem 3 is to show the equivalence of two asymptotic formulas

$$\int_0^T g(t, T)dt \sim T$$

and

$$\int_0^\infty g(t, T) \left(\frac{\sin t/2T}{t} \right)^2 dt \sim \frac{\pi}{2} \frac{1}{2T},$$

where

$$g(t, \eta) = 2\mathrm{Re} \prod_{n=1}^{N} \frac{\zeta'}{\zeta} \left(\frac{1}{2} + \frac{C + A_n}{\log \eta} + i\varepsilon_n t \right) \Big/ \left(f(C, \mathbf{A}, J) \log^N \eta \right).$$

Proof of Theorem 3

We appeal to modified versions of two Lemmas in Goldston's paper. These concern the equivalence under certain conditions of

$$\int_0^T g(t,\eta)dt \sim T \tag{1}$$

as $T \to \infty$, and

$$\int_0^\infty g(t,\eta)\left(\frac{\sin \kappa t}{t}\right)^2 dt \sim \frac{\pi}{2}\kappa \tag{2}$$

as $\kappa \to 0+$.

Proof of Theorem 3

We appeal to modified versions of two Lemmas in Goldston's paper. These concern the equivalence under certain conditions of

$$\int_0^T g(t,\eta)dt \sim T \tag{1}$$

as $T \to \infty$, and

$$\int_0^\infty g(t,\eta)\left(\frac{\sin \kappa t}{t}\right)^2 dt \sim \frac{\pi}{2}\kappa \tag{2}$$

as $\kappa \to 0+$. First we see $1 \to 2$.

Lemma $1 \to 2$

Let $g(t,\eta)$ be a continuous function of t and η for $t \geq 0$ and $\eta \geq 2$.
Suppose that $g(t,\eta) \ll \log^N(t+2)$ and that $\int_0^T |g(t,\eta)|^2 dt \ll T$ holds for $\eta \log^{-N-1}\eta \leq T \leq \eta \log^{N+1}\eta$.
If (1) holds uniformly for $\eta \log^{-N-1}\eta \leq T \leq \eta \log^{N+1}\eta$, then (2) holds for $\eta \approx 1/\kappa$.

Proof of Theorem 3

To apply Lemma 1 $\rightarrow$ 2, we should prove that

$$\int_0^T g(t,T)dt \sim T$$

implies

$$\int_0^T g(t,\eta)dt \sim T$$

holds uniformly for $\eta \log^{-N-1} \eta \leq T \leq \eta \log^{N+1} \eta$.

Proof of Theorem 3

To apply Lemma 1 $\rightarrow$ 2, we should prove that

$$\int_0^T g(t, T)dt \sim T$$

implies

$$\int_0^T g(t, \eta)dt \sim T$$

holds uniformly for $\eta \log^{-N-1} \eta \leq T \leq \eta \log^{N+1} \eta$. It is proved by

Lemma

Assume RH, Hypothesis AC, and Hypothesis LC. Let $B_1, \ldots, B_N$ be fixed positive real numbers. Suppose that $\eta \log^{-N-2} \eta \leq T \leq \eta \log^{N+2} \eta$. Then we have
$\int_0^T \prod_{n=1}^N \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{B_n}{\log \eta} + i\varepsilon_n t\right) dt =$
$\int_0^T \prod_{n=1}^N \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{B_n}{\log T} + i\varepsilon_n t\right) dt + O(T \log^{N-1} T \ \log\log T).$

Proof of Theorem 3

Lemma

Assume RH, Hypothesis AC, and Hypothesis LC. Let $B_1, \ldots, B_N$ be fixed positive real numbers. Suppose that $\eta \log^{-N-2} \eta \leq T \leq \eta \log^{N+2} \eta$. Then we have
$\int_0^T \prod_{n=1}^N \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{B_n}{\log \eta} + i\varepsilon_n t\right) dt =$
$\int_0^T \prod_{n=1}^N \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{B_n}{\log T} + i\varepsilon_n t\right) dt + O(T \log^{N-1} T \ \log\log T).$

We use the fact

$$\log \eta = \log T + O(\log\log T)$$

and the following consequence of Corollary 1 : Under RH, AC and LC we have

$$\int_0^T \left| \prod_{j \leq J} \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{B_j}{\log T} + it\right) \right|^2 dt \ll T(\log T)^{2J}.$$

Proof of Theorem 3

For the converse, we need the followings:

$$\int_0^T g(t,\eta)dt \sim T \tag{3}$$

as $T \to \infty$, and

$$\int_0^\infty g(t,\eta)\left(\frac{\sin \kappa t}{t}\right)^2 dt \sim \frac{\pi}{2}\kappa \tag{4}$$

as $\kappa \to 0+$.

Lemma 4 $\to$ 3

Let $g(t,\eta)$ be a continuous function of t and η for $t \geq 0$ and $\eta \geq 2$. Suppose that $g(t,\eta) \ll \log^N(t+2)$ and that $\int_0^T |g(t,\eta)|^2 dt \ll T$ holds for $\eta \log^{-N-1}\eta \leq T \leq \eta \log^{N+1}\eta$.
Conversely, if (4) holds uniformly for $\eta^{-1}\log^{-N-1}\eta \leq \kappa \leq \eta^{-1}\log^{N+1}\eta$, then (3) holds for $\eta \approx T$.

Proof of Theorem 3

We also need

Lemma

Assume RH, Hypothesis AC, and Hypothesis LC. Let $B_1, \dots, B_N$ be fixed positive real numbers. Suppose that $\eta \log^{-N-2} \eta \le T \le \eta \log^{N+2} \eta$. Then we have

$$\int_0^\infty \prod_{n=1}^N \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{B_n}{\log \eta} + i\varepsilon_n t\right) \left(\frac{\sin t/2T}{t}\right)^2 dt \quad =$$

$$\int_0^\infty \prod_{n=1}^N \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{B_n}{\log T} + i\varepsilon_n t\right) \left(\frac{\sin t/2T}{t}\right)^2 dt + O(T^{-1} \log^{N-1} T \, \log\log T).$$

By the similar argument to $1 \to 2$, we can complete the proof of Theorem 3.

THANK YOU.